Giving Byes to Top Seeds Is Unfair

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The ITTF (International Table Tennis Federation) rules for single elimination draws may be briefly described as follows:

- Place the first two seeds in different halves, then randomly place the next two seeds in empty quarters, then randomly place the next four seeds in empty eighths, etc., until all seeds are placed.
- If byes are needed, give them to the top seeds in order.
- Randomly place the unseeded players.
- When placing players, place players from the same association in different sections of the draw.

We show that giving byes to the top seeds in order can lead to situations where a player would prefer to be seeded lower rather than higher.

Suppose there are seven players, p_i, $i = 1 \dots 7$, each from a different association. Assume that the probability that p_1 defeats any of the other players is x, where $\frac{1}{2} < x < 1$, and that all the other players are equal in ability. Assume that all players are seeded and that player p_i is seeded ith.

To make the draw according to the ITTF rules, place p_1 in the top slot and give him a bye in the first round. Place p_2 in the bottom slot. Draw p_3 and p_4 randomly into the two middle slots. Finally, draw the remaining three players randomly into the remaining three slots.

Let s_i be the player in slot i, where we number slots according to the way that people make a deterministic draw. I.e., s_1 is the top slot, s_2 is the bottom slot, s_3 is the slot at the top of the bottom half, etc. Straightforward calculation gives the following probabilities that the player in each slot will win the tournament.

Slot	Prob. Wins Tournament
s_1	x^2
s_4, s_5	$\frac{1-x}{4}$
s_2, s_3, s_6, s_7	$\frac{1-x}{4}\left(x + \frac{1}{2}\right)$

From this it is straightforward to calculate the probability (before the draw is made) that each player will win the tournament:

Player	Prob. Wins Tournament
p_1	x^2
p_2	$\frac{1-x}{4}\left(x+\frac{1}{2}\right)$
p_3, p_4	$\frac{1-x}{8}\left(x+\frac{3}{2}\right)$
p_5, p_6, p_7	$\frac{1-x}{6}(x+1)$

It is easy to check that $\frac{1}{2} < x < 1$ implies that

$$\frac{1-x}{8}\left(x+\frac{3}{2}\right) < \frac{1-x}{6}(x+1).$$

So, it is better to be seeded 5th–7th rather than 3rd–4th.

Here are the probabilities for various values of x that each player wins the tournament:

Player	x			
	0.6	0.7	0.8	0.9
p_1	0.3600	0.4900	0.6400	0.8100
p_2	0.1100	0.0900	0.0650	0.0350
p_3, p_4	0.1050	0.0825	0.0575	0.0300
p_5, p_6, p_7	0.1067	0.0850	0.0600	0.0317

The intuition is that it is better to be in the bottom half (away from the first seed). The 3rd and 4th seeds each have a $\frac{1}{2}$ chance of being in the bottom half, but the 5th–7th seeds each have a $\frac{2}{3}$ chance of being in the bottom. Therefore it is better to be seeded lower.

A fair way to make the draw is to place byes randomly. Seeding all players and placing byes randomly gives "cohort randomized seeding", which is always fair. This is proved in the article, "What Is the Correct Way to Seed a Knockout Tournament?", by Allen J. Schwenk, *The American Mathematical Monthly*, vol. 7, no. 2, Feb. 2000, pp. 140–150.

For the above example, we can see this intuitively. The farther away you are from the first seed, the better (since he may lose before reaching you). The 3rd and 4th seeds each have a $\frac{1}{2}$ chance of being in the top half, as do the 5th–7th seeds. However, the 5th–7th seeds have a $\frac{1}{4}$ chance of being in the top quarter, while the 3rd–4th seeds can't be in the top quarter. Hence, it is better to be seeded 3–4. To check this rigorously, you could, without too much trouble, calculate the probabilities assuming the bye is placed randomly. There are three more cases to consider (corresponding to each of the possible locations for the bye).